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ON THE OBLIQUE DERIVATIVE PROBLEM FOR DIFFUSION PROCESSES AND DIFFUSION EQUATIONS WITH HÖLDER CONTINUOUS COEFFICIENTS

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ABSTRACT. On a C^2-domain in a Euclidean space, we consider the oblique derivative problem for a diffusion equation and assume the coefficients of the diffusion and boundary operators are Hölder continuous. We then prove the uniqueness of diffusion processes and fundamental solutions corresponding to the problem. For the purpose, obtaining a stochastic representation of some solutions to the problem plays a key role; in our situation, a difficulty arises from the absence of a fundamental solution with C^2-smoothness up to the boundary. It is overcome by showing some stability of a fundamental solution and a diffusion process, respectively, under approximation of the domain. In particular, the stability of the fundamental solution is verified through construction: it is done by applying the parametrix method twice to a parametrix with explicit expression.

1. INTRODUCTION

Diffusion processes with boundary conditions on smooth domains in a Euclidean space or a manifold have been constructed by several methods or in several frameworks: by analytic methods (cf. [3], [24], [29], [30]), by using stochastic differential equations (cf. [16], [26], [35], [21], [22]), in the martingale formulation (cf. [28], [1], [20], [15]) and by means of Poisson point processes of Brownian excursions (cf. [36], [31]). In the case of not necessarily smooth domains, the Skorohod problem approach is useful for constructing diffusion processes with reflection (cf. [32], [19], [9], [23], [5], [6], [4], [7]). In particular, we refer to [4] for a general existence result of such diffusion processes and to [5], [6], [7] which have a close relationship to the subject of the present paper.

In this paper, we shall focus on diffusion processes with oblique reflection on smooth domains and treat them in the martingale formulation (see [28], [15]). Such a process on a C^2-domain D in the Euclidean d-space R^d is expected to be characterized by a second-order parabolic differential operator $\mathscr{A}$ on $[0, \infty) \times D$ and a first-order differential operator $\mathscr{B}$ on $[0, \infty) \times \partial D$:

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$$\mathscr{A} \equiv \mathscr{A}(s, x; \partial_s, \partial_x) := \partial/\partial s + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s, x)\partial^2/\partial x_i \partial x_j + \sum_{i=1}^{d} b_i(s, x)\partial/\partial x_i + c(s, x),$$

$$\mathscr{B} \equiv \mathscr{B}(s, x; \partial_x) := \sum_{i=1}^{d} \beta_i(s, x)\partial/\partial x_i + \gamma(s, x).$$

Throughout this paper, we assume that the coefficients of $\mathscr{A}$ and $\mathscr{B}$ are bounded, the matrix $(a_{ij}(s, x))$ is symmetric, and $c(s, x)$ and $\gamma(s, x)$ are nonpositive. Moreover we assume $\mathscr{A}$ and $\mathscr{B}$ are nondegenerate: the diffusion matrix $a(s, x) = (a_{ij}(s, x))$ is positive definite and the vector field $\beta(s, x) = (\beta_i(s, x))$ is in the inward direction. Let

$$\mathscr{L} \equiv \mathscr{L}(s, x; \partial_s, \partial_x) := 1_D(x)\mathscr{A}(s, x; \partial_s, \partial_x) + 1_{\partial D}(x)\mathscr{B}(s, x; \partial_x),$$

where $1_A(x)$ denotes the indicator of a set A. Under Hölder continuity of $a_{ij}(s, x)$ and $\beta_i(s, x)$, we first prove the uniqueness of solutions to the martingale problem and the coupled martinagle problem for $\mathscr{L}$ with $c = \gamma = 0$ (Theorem 2.5), and next prove the uniqueness of fundamental solutions to the terminal value problem for $\mathscr{L} = 0$ (Theorem 2.8).

Our key tool for proving the first result is some stability of a fundamental solution to the terminal value problem for $\mathscr{L} = 0$ on the upper half space under approximation of the domain (see Theorem 4.1); restricting the domain to the upper half space does not lose the generality because the localization argument can be applied (see [28]). Although the authors of [2] and [12] have constructed fundamental solutions on general domains, it is hard to verify the stability through the construction of the fundamental solutions. Hence we give another way of constructing a fundamental solution on the upper half space. The uniqueness of solutions to the coupled martingale problem is used to obtain the second result.

The main results (Theorems 2.5 and 2.8) are stated in §2. Section 3 is devoted to the construction of a fundamental solution on the upper half space (see Theorem 3.3). In §4, we discuss the stability of the fundamental solution constructed in §3 (see Theorem 4.1). In §5, we prove Theorems 2.5 and 2.8. We give some additional remarks in §6.

2. The statement of the main results

First we shall give some notation and definitions used later. We call a function $\Phi \in C_b^2(R^d)$ a *defining function* of an open set D in R^d if $D = \{x : \Phi(x) > 0\}$, $\partial D = \{x : \Phi(x) = 0\}$ and $|\nabla\Phi(x)| \geq 1$ for $x \in \partial D$.

Now we define the martingale problem for $\mathscr{L}_0$, where $\mathscr{L}_0 \equiv 1_D\mathscr{A}_0 + 1_{\partial D}\mathscr{B}_0$ denotes the operator $\mathscr{L}$ with $c = \gamma = 0$. When we consider the martingale problem, we assume that the domain D has a defining function Φ. Let $W = C([0, \infty) \to R^d)$ and, for $w \in W$, let $X(t, w) = w(t)$. Denote by $\mathscr{W}_t^s$ the σ-field generated by $X(u)$ $(s \leq u \leq t)$ and let $\mathscr{W} = \mathscr{W}_\infty^0$. Moreover, we set $\widehat{W} := C([0, \infty) \to \overline{D})$.

Definition 2.1. We say that a probability measure P on $(W, \mathscr{W})$ is a solution to the martingale problem for $\mathscr{L}_0$ starting at (s, x) $(s \geq 0, x \in \overline{D})$ if

(i) $P(X(s) = x) = 1$;
(ii) $P(\widehat{W}) = 1$;
(iii) there exists a $\{\mathscr{W}_t^s\}$-adapted continuous increasing process $\{l(t)\}$ such that

$$E^P[l(t)] < +\infty \quad \text{for any } t \geq s,$$

$$l(t) = \int_s^t 1_{\partial D}(X(u))\, dl(u) \quad \text{for any } t \geq s \ (P\text{-a.s.}),$$

and for every $f \in C_b^{1,2}([0, \infty) \times \overline{D})$

$$M_f(t) := f(t, X(t)) - f(s, X(s)) - \int_s^t \mathscr{A}_0 f(u, X(u))\, du - \int_s^t \mathscr{B}_0 f(u, X(u))\, dl(u)$$

is a P-martingale with respect to the filtration $\{\mathscr{W}_t^s\}$.

The equivalence between the martingale problem and the submartingale problem for $\mathscr{L}_0$ is obtained by Theorem 2.4 of [28] and Theorem I.2 of [15]. That is, the following holds.

Proposition 2.2. *Assume that for some $\delta > 0$*

$$\beta(s, x) \cdot \nabla\Phi(x) \geq \delta \quad \textit{for all } (s, x) \in [0, \infty) \times \partial D.$$

Then a probability measure P on $(W, \mathscr{W})$ is a solution to the martingale problem for $\mathscr{L}_0$ if and only if P is a solution to the submartingale problem for $\mathscr{L}_0$.

Next we consider the coupled martingale problem for $\mathscr{L}_0$. Let

$$V = C([0, \infty) \to R)$$

and $U = W \times V$. Assume that W, V and U each are equipped with the locally uniform convergence topology. For $v \in V$, set $L(t, v) := v(t)$. Denote by $\mathscr{U}_t^s$ the σ-field generated by $(X(u), L(u))$ $(s \leq u \leq t)$ and let $\mathscr{U} = \mathscr{U}_\infty^0$. If we set $\widetilde{V} = \{L \in V : L \text{ is increasing}\}$, $\widetilde{V}$ is a closed subset of V.

Definition 2.3. We say that a probability measure $\overline{P}$ on $(U, \mathscr{U})$ is a solution to the coupled martingale problem for $\mathscr{L}_0$ starting at (s, x) $(s \geq 0, x \in \overline{D})$ if

(i) $\overline{P}(X(s) = x, L(s) = 0) = 1$;
(ii) $\overline{P}(\widehat{W} \times \widetilde{V}) = 1$;
(iii) $E^{\overline{P}}[L(t)] < +\infty$ for any $t \geq s$,

$$L(t) = \int_s^t 1_{\partial D}(X(u))\, dL(u) \quad \text{for any } t \geq s \ (\overline{P}\text{-a.s.}),$$

and for every $f \in C_b^{1,2}([0, \infty) \times \overline{D})$

$$\overline{M}_f(t) := f(t, X(t)) - f(s, X(s)) - \int_s^t \mathscr{A}_0 f(u, X(u))\, du - \int_s^t \mathscr{B}_0 f(u, X(u))\, dL(u)$$

is a $\overline{P}$-martingale with respect to the filtration $\{\mathscr{U}_t^s\}$.

As described in Remark 1 in [15], we have

Proposition 2.4. *If P is a solution to the martingale problem for $\mathcal{L}_0$, then the probability measure $\widetilde{P}$ on $(U, \mathcal{U})$ induced by the process $\{X(t), l(t)\}$ is a solution to the coupled martingale problem for $\mathcal{L}_0$.*

We set the following assumption for the coefficients of $\mathcal{L}$.

(A-1) (i) a_{ij} $(i, j = 1, 2, \dots, d)$ are bounded $(\alpha/2, \alpha)$-Hölder continuous real functions defined on $[0, \infty) \times \overline{D}$, and b_i $(i = 1, 2, \dots, d)$ and c are bounded continuous real functions defined on $[0, \infty) \times \overline{D}$.

(ii) $a(s, x)$ is positive definite for each $(s, x) \in [0, \infty) \times \overline{D}$.

(iii) β_i $(i = 1, 2, \dots, d)$ are bounded $(\alpha/2, \alpha)$-Hölder continuous real functions defined on $[0, \infty) \times \partial D$, and γ is a bounded continuous function defined on $[0, \infty) \times \partial D$.

(iv) There exists a positive constant δ such that

$$\beta(s, x) \cdot \nabla\Phi(x) \geq \delta \quad \text{for } (s, x) \in [0, \infty) \times \partial D.$$

Here a function $f(s, x)$ defined on $[0, \infty) \times \overline{D}$ (resp. $[0, \infty) \times \partial D$) is called $(\alpha/2, \alpha)$-Hölder continuous $(0 < \alpha \leq 1)$ if there exists a positive constant K such that

$$|f(x, s) - f(s', x')| \leq K\{|s - s'|^{\alpha/2} + |x - x'|^{\alpha}\}$$

for every $(s, x), (s', x') \in [0, \infty) \times \overline{D}$ (resp. $[0, \infty) \times \partial D$).

The first main result is

Theorem 2.5. *Under the assumption* (A-1), *the martingale problem and the coupled martingale problem for $\mathcal{L}_0$ each have a unique solution for any starting point $(s, x) \in [0, \infty) \times \overline{D}$. Furthermore the diffusion process defined by the solutions to the martingale problem has a transition density.*

Next we state the notion of a fundamental solution to the terminal value problem for $\mathcal{L} = 0$.

Definition 2.6. Let D be a C^2-domain. A function $p(s, x; t, y)$ $(0 \leq s < t; x, y \in \overline{D})$ is called a fundamental solution to the terminal value problem for $\mathcal{L}(s, x; \partial_s, \partial_x) = 0$ if

(i) $p(s, x; t, y)$ is measurable;

(ii) for each $t > 0$ and $y \in \overline{D}$

$$P(\cdot, \cdot; t, y) \in C^{1,2}([0, t) \times D) \cap C^{0,1}([0, t) \times \overline{D})$$

and

$$\mathcal{L}(s, x; \partial_s, \partial_x)p(s, x; t, y) = 0 \quad \text{for } (s, x) \in [0, t) \times \overline{D};$$

(iii) for each $t > 0$ and $\varepsilon > 0$

$$\sup_{(s,x)\in[0,t)\times\overline{D}} \int_{\overline{D}} |p(s, x; t, y)|\, dy < +\infty,$$

$$\sup_{(s,x)\in[0,t-\varepsilon)\times\overline{D}} \int_{\overline{D}} |\partial_x p(s, x; t, y)|\, dy < +\infty,$$

$$\sup_{(x,s)\in[0,t-\varepsilon)\times\overline{D}_\varepsilon} \int_{\overline{D}} |\partial_s^m \partial_x^n p(s, x; t, y)|\, dy < +\infty$$

provided $2m + n = 2$, where $\overline{D}_\varepsilon = \{x \in D : d(x, \partial D) \geq \varepsilon\}$;

(iv) for every $f \in C_b(\overline{D})$

$$\lim_{s \uparrow t} \int_{\overline{D}} p(s, x; t, y) f(y)\, dy = f(x) \quad \text{for } x \in \overline{D}.$$

Remark. It is clear that each Green's function constructed in [2] and [12] becomes a fundamental solution in the sense of Definition 2.6.

We impose an assumption on the domain D.

(A-2) Each of the connected components of the boundary is a C^2-hypersurface, and the domain D satisfies a uniform interior sphere condition and a uniform exterior sphere condition, that is, there exists a positive constant r such that for each $x \in \partial D$ we can find two balls B_1 and B_2 satisfying the conditions:

(1) the radius of B_i is greater then r $(i = 1, 2)$;
(2) $B_1 \subset D^c$ and $B_2 \subset D$;
(3) $\overline{B}_1 \cap \overline{D} = \{x\}$ and $\overline{B}_2 \cap D^c = \{x\}$.

Remark. Assuming both the uniform sphere conditions is equivalent to doing the existence of a tubular neighborhood of the boundary with uniform thickness. If the boundary ∂D is compact and each of its connected components is a C^2-hypersurface, then the domain D automatically satisfies both the uniform sphere conditions.

Proposition 2.7. *Assume that the domain D satisfies the assumption* (A-2). *Then D has a defining function Φ with $|\nabla\Phi(x)| = 1$ for every $x \in \partial D$.*

Proof. Let r be the constant which appears in the condition (A-2). Define

$$\Phi_0(x) = \begin{cases} d(x, \partial D) & \text{for } x \in D, \\ 0 & \text{for } x \in \partial D, \\ -d(x, \partial D) & \text{for } x \in D^c. \end{cases}$$

Then, from the proof of the result of [13, Appendix] or [14, Appendix B], it follows that Φ_0 is of C^2-class in $\{x \in R^d : d(x, \partial D) < r\}$, the derivatives up to the second order are bounded near the boundary ∂D and $|\nabla\Phi_0(x)| = 1$ for every $x \in \partial D$. Using a partition of unity, we can construct the desired function Φ. □

The second main result is

Theorem 2.8. *Suppose that* (A-1) *and* (A-2) *are satisfied. Then for any fundamental solution $p(s, x; t, y)$ the following equalities hold*:

$$\begin{aligned} &\int_{\overline{D}} p(s, x; t, y) f(y)\, dy \\ &\quad = \overline{E}_{s,x}\left[f(X(t)) \exp\left\{\int_s^t c(u, X(u))\, du + \int_s^t \gamma(u, X(u))\, dL(u)\right\}\right] \\ &\quad = E_{s,x}\left[f(X(t)) \exp\left\{\int_s^t c(u, X(u))\, du + \int_s^t \gamma(u, X(u))\, dl(u)\right\}\right] \end{aligned}$$

for every $f \in C_b(\overline{D})$, *where* $\overline{E}_{s,x}$ (*resp.* $E_{s,x}$) *denotes the expectation with respect to the unique solution* $\overline{P}_{s,x}$ (*resp.* $P_{s,x}$) *to the coupled martingale problem* (*resp. the martingale problem*) *for* $\mathscr{L}_0$ *starting at* $(s, x) \in [0, \infty) \times \overline{D}$.

Remark. If we further assume that $p(s,x;t,\cdot)$ is continuous, then $p(s,x;t,y)$ is uniquely determined and nonnegative.

Corollary. *Given* $t > 0$, *let* $u(s, x) \in C_b([0, t] \times \overline{D}) \cap C_b^{0,1}([0, t - \varepsilon] \times \overline{D}) \cap C^{1,2}([0, t) \times D)$ *for each* $\varepsilon > 0$ *and*

$$\mathscr{L}(s, x; \partial_s, \partial_x)u(s, x) = 0 \quad \textit{for } (s, x) \in [0, t) \times \overline{D}.$$

Then $u(s, x)$ *has the representation*

$$u(s, x) = \int_{\overline{D}} p(s, x; t, y)u(t, y)\,dy \qquad ((s, x) \in [0, t] \times \overline{D}),$$

provided a fundamental solution $p(s, x; t, y)$ *exists.*

3. Construction of a fundamental solution on the upper half space

In this section, we take D and $\mathscr{B}$ as in the form:

$$D = \{x = (x_1, \dots, x_d) : x_d > 0\},$$

$$\mathscr{B} \equiv \mathscr{B}(s, x; \partial_x) = \partial/\partial\nu(s, x) + \sum_{I=1}^{d-1} \mu_i(s, x)\partial/\partial x_i + \gamma(s, x),$$

where $\partial/\partial\nu(s, x) \equiv \frac{1}{2}\sum_{j=1}^{d} a_{dj}(s, x)\partial/\partial x_j$ is the conormal derivative with respect to the operator $\mathscr{A}$.

Throughout this section, we assume the following conditions:

(A-1′) (i) a_{ij}, b_j and c are bounded $(\alpha/2, \alpha)$-Höllder continuous real functions defined on $[0, \infty) \times \overline{D}$;

(ii) $(a_{ij}(s, x))$ is uniformly positive definite;

(iii) μ_i and γ are bounded $(\alpha/2, \alpha)$-Hölder continuous real functions on $[0, \infty) \times \partial D$.

1°. First we give a parametrix to the terminal value problem for the equation $\mathscr{L} = 0$ and investigate its property. For $\tau \geq 0$ and $\eta \in \overline{D}$, let

$$\widehat{\mathscr{A}} \equiv \widehat{\mathscr{A}}(\tau, \eta; \partial_s, \partial_x) := \partial/\partial s + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\tau, \eta)\partial^2/\partial x_i\partial x_j,$$

$$\widehat{\mathscr{B}} \equiv \widehat{\mathscr{B}}(\tau, \eta; \partial_x) = \partial/\partial\tilde{\nu}(\tau, \eta) + \sum_{i=1}^{d-1} \mu_i(\tau, \eta)\partial/\partial x_i,$$

where $\partial/\partial\tilde{\nu}(\tau, \eta) = \frac{1}{2}\sum_{j=1}^{d} a_{dj}(\tau, \eta)\partial/\partial x_j$. Furthermore let

$$\widehat{\mathscr{L}} \equiv \widehat{\mathscr{L}}(\tau, \eta; \partial_s, \partial_x) =:= 1_D(x)\widehat{\mathscr{A}}(\tau, \eta; \partial_s, \partial_x) + 1_{\partial D}(x)\widehat{\mathscr{B}}(\tau, \eta; \partial_x).$$

We call the fundamental solution $p^{\tau,\eta}(s, x; t, y)$ to the terminal value problem for $\widehat{\mathscr{L}} = 0$ a *parametrix* to the terminal value problem for $\mathscr{L} = 0$.

Before giving the precise expression of $p^{\tau,\eta}(s, x; t, y)$, we shall introduce some notation. Let $S = (s_{ij}(\tau, \eta))$ be the symmetric positive square root of

$(a_{ij}(\tau, \eta))$ and $s_i = s_i(\tau, \eta)$ the ith column vector of S $(i = 1, 2, \dots, d)$. Construct an orthonormal system $\{t_d, t_{d-1}, \dots, t_1\}$ from $\{s_d, s_{d-1}, \dots, s_1\}$ by the Gram-Schmidt orthogonalization and set $T := (t_1, t_2, \dots, t_d)$. If we set $U = ST$, then $U = (u_{ij})$ has the following form:

$$U = \begin{bmatrix} \widetilde{U} & \tilde{u}_d \\ 0 & u_{dd} \end{bmatrix},$$

where $\widetilde{U}$ is a $(d-1) \times (d-1)$-matrix and

$$u_d = \begin{bmatrix} \tilde{u}_d \\ u_{dd} \end{bmatrix}$$

is the dth column vector of U. Note that $u_{ij} = u_{ij}(\tau, \eta)$ $(i, j = 1, 2, \dots, d)$ are bounded $(\alpha/2, \alpha)$-Hölder continuous in (τ, η), $\widetilde{U}$ is nonsingular and $u_{dd} = \sqrt{a_{dd}(\tau, \eta)} > 0$. Define three $(d-1)$-column vectors $\tilde{\mu}$, $\tilde{\mu}'$ and $\tilde{\mu}''$ as follows:

$$\begin{aligned} \tilde{\mu} &\equiv \tilde{\mu}(\tau, \eta) := (\mu_i(\tau, \eta)), \\ \tilde{\mu}' &\equiv \tilde{\mu}'(\tau, \eta) := (\mu_i(\tau, \eta) + \tfrac{1}{2} a_{di}(\tau, \eta)), \\ \tilde{\mu}'' &\equiv \tilde{\mu}''(\tau, \eta) := 2a_{dd}(\tau, \eta)^{-1} \tilde{\mu}'(\tau, \eta). \end{aligned}$$

To simplify typography, henceforth, we shall assume that a point of R^d or R^{d-1} is indicated as a row vector, but in the matrix multiplication it acts as a column vector.

Let $g(t, u)$ be the one-dimensional Gauss kernel, that is,

$$g(t, u) = (2\pi t)^{-1/2} \exp\left\{-\frac{u^2}{2t}\right\}.$$

Define

$$\begin{aligned} h(t, u) &:= -\frac{\partial g}{\partial u}(t, u), \\ G(t, \tilde{x}) &:= \prod_{i=1}^{d-1} g(t, x_i), \end{aligned}$$

where $\tilde{x} = (x_1, x_2, \dots, x_{d-1}) \in R^{d-1}$.

Then the parametrix $p^{\tau, \eta}(s, x; t, y)$ is given by

$$p^{\tau, \eta}(s, x; t, y) = p_1^{\tau, \eta}(s, x; t, y) + p_2^{\tau, \eta}(s, x; t, y),$$

where

$$\begin{aligned} p_1^{\tau, \eta}(s, x; t, y) &:= G(t-s, \widetilde{U}^{-1}(\tilde{x} - \tilde{y} - u_{dd}^{-1}(x_d - y_d)\tilde{u}_d)) \\ &\qquad \times \{g(t-s, u_{dd}^{-1}(x_d - y_d)) - g(t-s, u_{dd}^{-1}(x_d + y_d))\} \det S^{-1}, \\ p_2^{\tau, \eta}(s, x; t, y) &:= 2u_{dd}^{-1} \int_0^\infty G(t-s, \widehat{U}^{-1}(\tilde{x} - \tilde{y} - u_{dd}^{-1}(x_d - y_d + l)\tilde{u}_d + l\tilde{\mu}'')) \\ &\qquad \times h(t-s, u_{dd}^{-1}(x_d + y_d + l)) \det S^{-1} \, dl, \end{aligned}$$

where $x = (x_1, x_2, \dots, x_d) = (\tilde{x}, x_d)$ (see [33] for derivation of the explicit formula for the parametrix). Of course, $\widetilde{U} = \widetilde{U}(\tau, \eta)$, $\tilde{u}_d = \tilde{u}_d(\tau, \eta)$, $u_{dd} =$

$u_{dd}(\tau, \eta)$, $S = S(\tau, \eta)$ and $\mu'' = \mu''(\tau, \eta)$. Then $p^{\tau,\eta}(s, x; t, y) > 0$ and, for each $\tau \geq 0$, $t > 0$ and $\eta, y \in \overline{D}$,

$$\widetilde{\mathscr{L}}(\tau, \eta; \partial_s, \partial_x)p^{\tau,\eta}(s, x; t, y) = 0 \qquad ((s, x) \in [0, t) \times \overline{D}),$$
$$\int_{\overline{D}} p^{\tau,\eta}(s, x; t, y)\, dy = 1,$$

and further we have the following estimates. In the following, by $\sigma(d\eta)$ we mean the surface element on ∂D, that is, $\sigma(d\eta) = d\tilde{\eta}\delta_0(d\eta_d)$ $(\eta = (\tilde{\eta}, \eta_d))$.

Proposition 3.1. *For nonnegative integers m, n, there exist positive constants K and C such that*

(i) $$|\partial_s^m \partial_x^n p^{\tau,\eta}(s, x; t, y)| \leq K(t-s)^{-(d+2m+n)/2} \exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $\tau \geq 0$, $0 \leq s < t$ and $\eta, x, y \in \overline{D}$;

(ii)
$$\begin{aligned}|\partial_x^n p^{\tau,\eta}(s, x; t, y) - \partial_x^n p^{\tau,\eta}(s, x'; t, y)| \\ \leq K|x-x'|(t-s)^{-(d+n+1)/2}\left[\exp\left\{-C\frac{|x-y|^2}{t-s}\right\} + \exp\left\{-C\frac{|x'-y|^2}{t-s}\right\}\right]\end{aligned}$$

for $\tau \geq 0$, $0 \leq s < t$ and $\eta, x, x', y \in \overline{D}$;

(iii)
$$\begin{aligned}|\partial_s^m \partial_x^n p^{\tau,\xi}(s, x; t, y) - \partial_s^m \partial_x^n p^{\tau,\eta}(s, x; t, y)| \\ \leq K|\xi-\eta|^\alpha (t-s)^{-(d+2m+n)/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}\end{aligned}$$

for $\tau \geq 0$, $0 \leq s < t$ and $\xi, \eta, x, y \in \overline{D}$;

(iv) $$\left|\int_D \partial_s^m \partial_x^n p^{\tau,\eta}(s, x; \tau, \eta)\, d\eta\right| \leq K(\tau-s)^{-(2m+n-\alpha)/2}$$

for $\tau \geq 0$, $0 \leq s < \tau$ and $x \in \overline{D}$ provided $2m+n \geq 1$;

(v) $$\left|\int_{\partial D} \frac{\partial p^{\tau,\eta}}{\partial x_j}(s, x; \tau, \eta)\sigma(d\eta)\right| \leq K(\tau-s)^{\alpha/2-1}\exp\left\{-C\frac{x_d^2}{\tau-s}\right\}$$

for $1 \leq j \leq d-1$, $\tau \geq 0$, $0 \leq s < \tau$ and $x \in \overline{D}$;

(vi) $$\left|\int_{\partial D} \frac{\partial p^{\tau,\eta}}{\partial x_d}(s, x; \tau, \eta)\sigma(d\eta)\right| \leq K(\tau-s)^{\alpha/2-1}$$

for $\tau \geq 0$, $0 \leq s < \tau$ and $x \in \partial D$.

Remark. The constant C in Proposition 3.1 depends only on the bound of the eigenvalues of the diffusion matrix $(a_{ij}(s, x))$. Therefore, the constants C in Theorems 3.3 and 4.1 below have the same dependency.

For the proof of Proposition 3.1, we need the following lemma.

Lemma 3.2. *Let* $\tilde{\delta} := \tilde{\mu}'' - u_{dd}^{-1}\tilde{u}_d$. *Then, for* $\varepsilon > 0$, *there exist positive constants* K *and* C *such that*
(3.1)

$$\int_0^\infty \exp\left\{-\varepsilon\frac{|\widetilde{U}^{-1}(\tilde{x}-\tilde{y}-u_{dd}^{-1}(x_d-y_d)\tilde{u}_d+l\tilde{\delta})|^2+|u_{dd}^{-1}(x_d+y_d+l)|^2}{2(t-s)}\right\}dl$$
$$\leq K(t-s)^{1/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $\tau \geq 0$, $0 \leq s < t$ *and* $\eta, x, y \in \overline{D}$.

Proof. For simplicity, we set $\varepsilon = 1$ and let

$$\tilde{\xi} := \tilde{x} - \tilde{y} - u_{dd}^{-1}(x_d - y_d)\tilde{u}_d.$$

If we make the substitution $\rho = u_{dd}^{-1}(x_d + y_d + l)/\sqrt{t-s}$, then the left-hand side of (3.1), say I, is given by

$$u_{dd}\sqrt{t-s}\int_{\rho_0}^\infty \exp\left\{-\frac{|\widetilde{U}(\tilde{\xi}-(x_d+y_d)\tilde{\delta})|^2+|u_{dd}^{-1}\sqrt{t-s}\,\rho\tilde{\delta}|^2+(t-s)\rho^2}{2(t-s)}\right\}d\rho,$$

where $\rho_0 = u_{dd}^{-1}(x_d + y_d)\sqrt{t-s}$. We define

$$\begin{aligned}
\widetilde{W} &:= \widetilde{U}\widetilde{U}^* \text{ (the symbol $*$ denotes the transposition of matrix)},\\
a &:= \{(1+u_{dd}^2\widetilde{W}^{-1}\tilde{\delta}\cdot\tilde{\delta})/2\}^{1/2},\\
\lambda &:= \{u_{dd}\widetilde{W}^{-1}(\tilde{\xi}-(x_d+y_d)\tilde{\delta})\cdot\tilde{\delta}\}/\{2a\sqrt{t-s}\},\\
\kappa &:= [\widetilde{W}^{-1}(\tilde{\xi}-(x_d+y_d)\tilde{\delta})\cdot(\tilde{\xi}-(x_d+y_d)\tilde{\delta})/\{2(t-s)\}]^{1/2}.
\end{aligned}$$

Then

$$\begin{aligned}
&[|\widetilde{U}^{-1}(\tilde{\xi}-(x_d+y_d)\tilde{\delta})|^2+|u_{dd}\sqrt{t-s}\,\rho\tilde{\delta}|^2+(t-s)\rho^2]/\{2(t-s)\}\\
&\quad=(a\rho+\lambda)^2-\lambda^2+\kappa^2.
\end{aligned}$$

Therefore, if we make the substitution $\theta = a\rho + \lambda$ and define $\chi := \rho_0 + \lambda$, then

$$I = a^{-1}u_{dd}\sqrt{t-s}\exp(\lambda^2-\kappa^2)\int_\chi^\infty \exp(-\theta^2)\,d\theta.$$

First we consider the case $\chi \geq 0$. Then the inequality

$$\int_\chi^\infty \exp(-\theta^2)\,d\theta \leq \exp(-\chi^2)$$

implies

$$I \leq a^{-1}u_{dd}\sqrt{t-s}\exp(\lambda^2-\kappa^2-\chi^2).$$

Now

$$\begin{aligned}
\lambda^2-\kappa^2-\chi^2 &= \{\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\xi}+u_{dd}^{-2}(x_d+y_d)^2\}/(2(t-s)\}\\
&\geq \{\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\xi}+u_{dd}^{-2}(x_d-y_d)^2\}/\{2(t-s)\}\\
&= |U^{-1}(x-y)|^2/\{2(t-s)\}\\
&= |S^{-1}(x-y)|^2/\{2(t-s)\}\\
&\geq C|x-y|^2/(t-s)
\end{aligned}$$

for some positive constant C; hence

$$I \le a^{-1} u_{dd} \sqrt{t-s} \exp\left\{-C\frac{|x-y|^2}{t-s}\right\}.$$

Next we consider the case $\chi < 0$. Note that

$$\chi < 0 \quad \text{iff} \quad -\widetilde{W}\tilde{\xi} \cdot \tilde{\delta} > u_{dd}^{-2}(x_d + y_d).$$

On the other hand,

$$\begin{aligned}\lambda^2 - \kappa^2 = & -\{4a^2(t-s)\}^{-1}[\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\xi} + \{-2(x_d+y_d)\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\delta}\} \\ & + u_{dd}^2\{(\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\xi})(\widetilde{W}^{-1}\tilde{\delta}\cdot\tilde{\delta}) - (\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\delta})^2\} + (x_d+y_d)^2\widetilde{W}^{-1}\tilde{\delta}\cdot\tilde{\delta}] \\ \le & -\{4a^2(t-s)\}^{-1}\{\widetilde{W}^{-1}\tilde{\xi}\cdot\tilde{\xi} + u_{dd}^2(x_d+y_d)^2\} \\ \le & -C|x-y|^2/(t-s)\end{aligned}$$

for some positive constant C; here, for simplicity, we use the same letter C as above. Hence

$$\begin{aligned} I &\le a^{-1}u_{dd}\sqrt{t-s}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}\int_{-\infty}^{\infty}\exp(-\theta^2)\,d\theta \\ &= \sqrt{\pi}a^{-1}u_{dd}\sqrt{t-s}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}.\end{aligned}$$

Consequently we complete the proof. □

Proof of Proposition 3.1. In the proof, we consider that each of the letters K and C may denote different constants in different places.

(i) For simplicity, we only treat the case $m = n = 0$. We first observe that

$$\begin{aligned} 0 \le p_1^{\tau,\eta}(s,x;t,y) &\le G(t-s, \widetilde{U}^{-1}(\tilde{x}-\tilde{y}-u_{dd}^{-1}(x_d-y_d)\tilde{u}_d)) \\ &\qquad \times g(t-s, u_{dd}^{-1}(x_d-y_d))\det S^{-1} \\ &\le K(t-s)^{-d/2}\exp\left\{-\frac{|\widetilde{U}^{-1}(\tilde{x}-\tilde{y}-u_{dd}^{-1}(x_d-y_d)\tilde{u}_d|^2 + |u_d^{-1}(x_d+y_d)|^2}{2(t-s)}\right\} \\ &\le K(t-s)^{-d/2}\exp\left\{-\frac{|U^{-1}(x-y)|^2}{2(t-s)}\right\} \\ &\le K(t-s)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}\end{aligned}$$

for some positive constants K and C.

Since u_{dd}^{-1} and $\det S^{-1}$ are bounded, we see that

$$\begin{aligned} 0 &\le p_2^{\tau,\eta}(s,x;t,y) \\ &\le K(t-s)^{-(d+1)/2}\int_0^{\infty} u_{dd}^{-1}(x_d+y_d+l)(t-s)^{-1/2} \\ &\quad \times \exp\left\{-\frac{|\widetilde{U}^{-1}(\tilde{x}-\tilde{y}-u_{dd}^{-1}(x_d-y_d)\tilde{u}_d+l\tilde{\delta})|^2 + |u_{dd}^{-1}(x_d+y_d+l)|^2}{2(t-s)}\right\}dl\end{aligned}$$

for some positive constant K. The fact

$$\sup_{\rho\ge 0}\rho^k\exp(-\varepsilon\rho^2) < +\infty \quad \text{for } k>0 \text{ and } \varepsilon > 0, \tag{3.2}$$

implies that for $0 < \varepsilon < 1$

$$\begin{aligned} p_2^{\tau,\eta}(s, x; t, y) &\le K(t-s)^{-(d+1)/2} \\ &\times \int_0^\infty \exp\left\{-(1-\varepsilon)\frac{|\tilde{U}^{-1}(\tilde{x}-\tilde{y}-u_{dd}^{-1}(x_d-y_d)\tilde{u}_d+l\tilde{\delta})|^2}{2(t-s)}\right. \\ &\qquad\qquad \left.+\frac{|u_{dd}^{-1}(x_d+y_d+l)|^2}{2(t-s)}\right\} dl \end{aligned}$$

with a positive constant K. Hence, from Lemma 3.2, we have

$$0 \le p_2^{\tau,\eta}(s, x; t, y) \le K(t-s)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}.$$

Therefore, the assertion (i) is proved.

(ii) As in the proof of Theorem 7 in [10, p. 17], we can prove the assertion by using (i) and the following fact: Let $0 < C' < C$ and $|x - x'| \le \sqrt{t-s}$. Then, for every x'' belonging to the line segment joining x and x',

$$\begin{aligned} &\exp\left\{-C\frac{|x''-y|^2}{t-s}\right\} \\ &\le \exp\left\{\frac{CC'}{C-C'}\right\}\left[\exp\left\{-C'\frac{|x-y|^2}{t-s}\right\} \wedge \exp\left\{C'\frac{|x'-y|^2}{t-s}\right\}\right], \end{aligned}$$

where $a \wedge b = \min\{a, b\}$.

(iii) For simplicity, we only prove the assertion in the case $m = n = 0$ for $p_2^{\tau,\eta}(s, x; t, y)$. Let

$$\begin{aligned} A &= |\tilde{U}^{-1}(\tau,\xi)(\tilde{x}-\tilde{y}-u_{dd}^{-1}(\tau,\xi)(x_d-y_d+l)\tilde{u}_d(\tau,\xi)+l\tilde{\mu}''(\tau,\xi))|^2 \\ &\qquad + |u_{dd}^{-1}(\tau,\xi)(x_d+y_d+l)|^2, \\ B &= |\tilde{U}^{-1}(\tau,\eta)(\tilde{x}-\tilde{y}-u_{dd}^{-1}(\tau,\eta)(x_d-y_d+l)\tilde{u}_d(\tau,\eta)+l\tilde{\mu}''(\tau,\eta))|^2 \\ &\qquad + |u_{dd}^{-1}(\tau,\eta)(x_d+y_d+l)|^2. \end{aligned}$$

Then

$$\begin{aligned} &p_2^{\tau,\xi}(s, x; t, y) - p_2^{\tau,\eta}(s, x; t, y) \\ &\quad = K(t-s)^{-(d+2)/2}\{u_{dd}^{-2}(\tau,\xi)\det S^{-1}(\tau,\xi) - u_{dd}^{-2}(\tau,\eta)\det S^{-1}(\tau,\eta)\} \\ &\quad\quad \times \int_0^\infty (x_d+y_d+l)\exp\left\{-\frac{A}{2(t-s)}\right\} dl \quad (= I_1) \\ &\quad\quad + Ku_{dd}^{-2}(\tau,\eta)\det S^{-1}(\tau,\eta)(t-s)^{-(d+2)/2} \\ &\quad\quad \times \int_0^\infty (x_d+y_d+l)\left[\exp\left\{-\frac{A}{2(t-s)}\right\} - \exp\left\{-\frac{B}{2(t-s)}\right\}\right] dl \quad (= I_2) \end{aligned}$$

with a positive constant K. Using the boundedness and $(\alpha/s, \alpha)$-Hölder continuity of $u_{dd}^{-2} \det S^{-1}$, Lemma 3.2 and (3.2), we see that

$$\begin{aligned}|I_1| &\le K|\xi-\eta|^{\alpha}(t-s)^{-(d+1)/2}\int_0^{\infty} u_{dd}^{-2}(\tau,\xi)(x_d+y_d+l)(t-s)^{-1/2} \\ &\qquad\qquad \times \exp\left\{-\frac{A}{2(t-s)}\right\} dl \\ &\le K|\xi-\eta|^{\alpha}(t-s)^{-(d+1)/2}\int_0^{\infty} \exp\left\{-(1-\varepsilon)\frac{A}{2(t-s)})\right\} dl \qquad (0<\varepsilon<1) \\ &\le K|\xi-\eta|^{\alpha}(t-s)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}\end{aligned}$$

for some positive constants K and C. The boundedness and $(\alpha/2, \alpha)$-Hölder continuity of $\widetilde{U}^{-1}$, $\tilde{u}_d$ and $\tilde{\delta}$ imply the inequality

$$|A-B| \le K|\xi-\eta|^{\alpha}\{|x-y|^2+(x_d+y_d+l)^2\}$$

with a positive constant K. Therefore, using the inequality

$$|\exp(-a)-\exp(-b)| \le \frac{1}{2}\{\exp(-a/2)+\exp(-b/2)\}|a-b| \qquad (a, b>0),$$

we have

$$\begin{aligned}|I_2| \le K(t-s)^{-(d+2)/2}\int_0^{\infty}(x_d+y_d+l)\left[\exp\left\{-\frac{A}{4(t-s)}\right\}+\exp\left\{-\frac{B}{4(t-s)}\right\}\right] \\ \times |\xi-\eta|^{\alpha}\{|x-y|^2+(x_d+y_d+l)^2\}(t-s)^{-1}\,dl\end{aligned}$$

with some positive constant K; hence, by Lemma 3.2 and (3.2),

$$|I_2| \le K|\xi-\eta|^{\alpha}(t-s)^{-d/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

with some positive constants K and C. Thus the assertion (iii) is proved.

The assertions (iv), (v) and (vi) follow, exactly in the same way as in [18, p. 357], from the equalities $\int_D p^{\tau,\xi}(s, x; \tau, \eta)\, d\eta = 1$,

$$\int_{\partial D}\frac{\partial p^{\tau,\xi}}{\partial x_j}(s, x; \tau, \eta)\sigma(d\eta)=0 \quad\text{and}\quad \int_{\partial D}\frac{\partial p^{\tau,\xi}}{\partial x_d}(s, x; \tau, \eta)\sigma(d\eta)=0,$$

respectively. □

2°. Now, using the idea of the parametrix method, we shall construct a fundamental solution to the terminal value problem for the equation $\mathscr{L}=0$. In what follows, we set

$$\begin{aligned}\overline{p}(s, x; t, y) &:= p^{t,y}(s, x; t, y), \\ \overline{q}(s, x; t, y) &:= \mathscr{A}(s, x; \partial_s, \partial_x)\overline{p}(s, x; t, y).\end{aligned}$$

Consider the integral equation for an unknown function $\phi(s, x; t, y)$ $(0 \le s < t, x \in D, y \in \overline{D})$:

$$\phi(s, x; t, y) = \overline{q}(s, x; t, y) + \int_s^t d\tau \int_D \overline{q}(s, x; \tau, \eta)\phi(\tau, \eta; t, y)\, d\eta.$$

Then, owing to Proposition 3.1, we can solve the equation by iteration and obtain necessary estimates for $\phi(s, x; t, y)$ in the same way as in Chapter 4 of [18]. Therefore, if we set

$$\tilde{p}(s, x; t, y) := \overline{p}(s, x; t, y) + \int_s^t d\tau \int_D \overline{p}(s, x; \tau, \eta)\phi(\tau, \eta; t, y)\, d\eta$$

$(0 \le s < t$ and $x, y \in \overline{D})$, then $\tilde{p}(s, x; t, y)$ satisfies the conditions (i), (ii):

(i) for each $t > 0$ and $y \in \overline{D}$, $\tilde{p}(\cdot, \cdot; t, y) \in C^{1,2}([0, t) \times \overline{D})$ and

$$\mathscr{A}(s, x; \partial_s, \partial_x)\tilde{p}(s, x; t, y) = 0 \quad \text{for } (s, x) \in [0, t) \times D;$$

(ii) for $T > 0$, there exist positive constants K and C such that

$$|\partial_s^m \partial_x^n \tilde{p}(s, x; t, y)| \le K(t-s)^{-(d+2m+n)/2} \exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $2m + n \le 2$, $0 \le s < t \le T$ and $x, y \in \overline{D}$.

Next define

$$\tilde{q}(s, x; t, y) := \mathscr{B}(s, x; \partial_x)\tilde{p}(s, x; t, y) \qquad (0 \le s < t, x \in \partial D, y \in \overline{D}).$$

Then it satisfies the following estimates:

(i) Given $T > 0$, there exist positive constants K and C such that

$$|\tilde{q}(s, x; t, y)| \le K(t-s)^{-(d+1-\alpha)/2} \exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $0 \le s < t \le T$, $x \in \partial D$ and $y \in \overline{D}$.

(ii) Given $T > 0$ and $\alpha' \in [0, \alpha)$, there exist positive constants K and C such that

$$|\tilde{q}(s, x; t, y) - \tilde{q}(s, x'; t, y)| \le K|x - x'|^{\alpha'}(t-s)^{-(d+1-\alpha+\alpha')/2}\left[\exp\left\{-C\frac{|x-y|^2}{t-s}\right\} + \exp\left\{-C\frac{|x'-y|^2}{t-s}\right\}\right]$$

for $0 \le s < t \le T$, $x, x' \in \partial D$ and $y \in \overline{D}$.

Consider the integral equation for an unknown function $\psi(s, x; t, y)$ $(0 \le s < t, x \in \partial D, y \in \overline{D})$:

$$\psi(s, x; t, y) = \tilde{q}(s, x; t, y) + \int_s^t d\tau \int_{\partial D} \tilde{q}(s, x; \tau, \eta)\psi(\tau, \eta; t, y)\sigma(d\eta).$$

We can also solve the equation by iteration and see that $\psi(s, x; t, y)$ has the same type estimates as $\tilde{q}(s, x; t, y)$. If we set

$$p(s, x; t, y) := \tilde{p}(s, x; t, y) + \int_s^t d\tau \int_{\partial D} \tilde{p}(s, x; \tau, \eta)\psi(\tau, \eta; t, y)\sigma(d\eta)$$

$(0 \le s < t$ and $x, y \in \overline{D})$, then $p(s, x; t, y)$ becomes a fundamental solution to the terminal value problem for $\mathscr{L} = 0$. That is

Theorem 3.3. *The function* $p(s, x; t, y)$ *satisfies the following*:

(i) $p(s, x; t, y)$ *is continuous.*

(ii) *For each* $t > 0$ *and* $y \in \overline{D}$,

$$p(\cdot, \cdot; t, y) \in C^{1,2}([0, t) \times D) \cap C^{0,1}([0, t) \times \overline{D})$$

and

$$\mathscr{L}(s, x; \partial_s, \partial_x)p(s, x; t, y) = 0 \quad \text{for } (s, x) \in [0, t) \times \overline{D}.$$

(iii) *Suppose that* $n = 0, 1$. *For each* $T > 0$, *there exist positive constants* K *and* C *such that*

$$|\partial_x^n p(s, x; t, y)| \le K(t-s)^{-(d+n)/2} \exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $0 \le s < t \le T$ *and* $x, y \in \overline{D}$.

(iv) *Suppose that* $2m + n = 2$. *Then, for given* $\rho > 1$ *and* $T > 0$, *there exist positive constants* K *and* C *such that*

$$|\partial_s^m \partial_x^n p(s, x; t, y)| \le K\{(t-s)^{-(d+2)/2} + x_d^{-\rho}(t-s)^{-(d+2-\alpha-\rho)}\} \times \exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $0 \le s < t \le T$, $x \in D$ *and* $y \in \overline{D}$.

(v) *For every bounded continuous function* f *defined on* $\overline{D}$,

$$\lim_{s \uparrow t} \int_{\overline{D}} p(s, x; t, y) f(y)\, dy = f(x)$$

boundedly in $x \in \overline{D}$.

Proof. A large part of the proof is carried out by the standard argument as in Chapter 4 of [18]; so it is omitted. We only note the following two points.

(1) In order to prove that $p(s, x; t, y)$ satisfies the boundary condition, we use the jump relation for $\tilde{p}(s, x; t, y)$: for $x_0 \in \partial D$

$$\begin{aligned}
\frac{\partial}{\partial x_d}\int_s^t d\tau \int_{\partial D} &\tilde{p}(s, x; \tau, \eta)\psi(\tau, \eta; t, y)\sigma(d\eta)|_{x=x_0} \\
= &\int_{(s+t)/2}^t d\tau \int_{\partial D} \frac{\partial \tilde{p}}{\partial x_d}(s, x_0; \tau, \eta)\psi(\tau, \eta; t, y)\sigma(d\eta) \\
&+ \int_s^{(s+t)/2} d\tau \int_{\partial D} \frac{\partial \tilde{p}}{\partial x_d}(s, x_0; \tau, \eta)\{\psi(\tau, \eta; t, y) - \psi(\tau, x_0; t, y)\}\sigma(d\eta) \\
&+ \int_s^{(s+t)/2} \psi(\tau, x_0; t, y)\, d\tau \int_{\partial D} \frac{\partial \tilde{p}}{\partial x_d}(s, x_0; \tau, \eta)\sigma(d\eta) \\
&- 2a_{dd}^{-1}(s, x_0)\psi(s, x_0; t, y).
\end{aligned}$$

This is reduced to verify the following fact

$$\begin{aligned}
\int_s^v &\psi(\tau, x_0; t, y) h(\tau - s, u_{dd}^{-1}(s, x_0)r)\, d\tau \\
&= \int_{w^2(v-s)^{-1}}^{\infty} \psi\left(s + \frac{w^2}{\rho}, x_0; t, y\right) \frac{1}{\sqrt{2\pi}} \rho^{-1/2} \exp\{-\rho/2\}\, d\rho \\
&\to \psi(s, x_0; t, y) \quad \text{as } r \downarrow 0,
\end{aligned}$$

where $w = u_{dd}^{-1}(s, x_0) r$.

(2) For the proof of (iv), we use the following inequalities

$$\begin{aligned}
&\left|\partial_s^m \partial_x^n \int_s^{(s+t)/2} d\tau \int_{\partial D} \tilde{p}(s, x; \tau, \eta)\psi(\tau, \eta; t, y)\sigma(d\eta)\right| \\
&\le K \int_s^{(s+t)/2} (\tau - s)^{-(d+2)/2}(t-\tau)^{-(d+1-\alpha)/2}\, d\tau \\
&\quad \times \int_{\partial D} \exp\left\{-C\frac{|x-\eta|^2}{\tau - s}\right\} \exp\left\{-C\frac{|\eta - y|^2}{t-\tau}\right\}\sigma(d\eta) \\
&\le K(t-s)^{-(d+1-\alpha)/2} \exp\left\{-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s}\right\} \exp\left\{-C\frac{y_d^2}{t-s}\right\} \\
&\quad \times \int_s^{(s+t)/2} (\tau - s)^{-3/2} \exp\left\{-C\frac{x_d^2}{\tau - s}\right\} d\tau \\
&\le K x_d^{-\rho}(t-s)^{-(d+1-\alpha)/2} \exp\left\{-C\frac{|\tilde{x}-\tilde{y}|^2}{t-s}\right\} \exp\left\{-C\frac{y_d^2}{t-s}\right\} \\
&\quad \times \int_s^{(s+t)/2} (\tau - s)^{(\rho-3)/2} \exp\left\{-C'\frac{x_d^2}{\tau - s}\right\} d\tau \qquad (0 < C' < C) \\
&\le K x_d^{-\rho}(t-s)^{-(d+2-\alpha-\rho)/2} \exp\left\{-C'\frac{|x-y|^2}{t-s}\right\}.
\end{aligned}$$

Remark. We may assert that, by more careful calculation, ρ in (iv) of Theorem 3.3 can be chosen in the interval $(0, 1)$; it is also verified by using a result of [2] or [12] and the uniqueness of fundamental solutions (see Remark stated below Theorem 2.8). If the coefficients β_i of the boundary operator $\mathscr{B}$ are more smooth, then ρ can be regarded as zero (cf. [8], [11]).

4. A STABILITY THEOREM FOR THE FUNDAMENTAL SOLUTION

We discuss the stability of the fundamental solution $p(s, x; t, y)$ constructed in §3. Therefore we use the same notation as in §3. Let us extend the domains of the coefficients of the operator $\mathscr{L}$ as follows. For $x = (\tilde{x}, x_d) \in R^d$ with $x_d < 0$, let $\underline{x} = (\tilde{x}, 0)$ and define

$$\underline{a}_{ij}(s, x) := a_{ij}(s, \underline{x}), \quad \underline{b}_i(s, x) := b_i(s, \underline{x}), \quad c(s, x) := c(s, \underline{x});$$
$$\underline{\mu}_i(s, x) := \mu_i(s, \underline{x}), \quad \underline{\gamma}(s, x) := \gamma(s, \underline{x});$$

$$\begin{aligned}
\underline{\mathscr{A}} \equiv \underline{\mathscr{A}}(s, x; \partial_s, \partial_x) &:= \partial/\partial s + \frac{1}{2}\sum_{i,j=1}^d \underline{a}_{ij}(s, x)\partial^2/\partial x_i \partial x_j \\
&+ \sum_{i=1}^d \underline{b}_i(s, x)\partial/\partial x_i + \underline{c}(s, x);
\end{aligned}$$

$$\underline{\mathscr{B}} \equiv \underline{\mathscr{B}}(s, x; \partial_x) = \partial/\partial\underline{\nu}(s, x) + \sum_{i=1}^{d-1} \underline{\mu}_i(s, x)\partial/\partial x_i + \underline{\gamma}(s, x),$$

where $\partial/\partial\underline{\nu}(s, x) = \frac{1}{2}\sum_{j=1}^{d} \underline{a}_{dj}(s, x)\partial/\partial x_j$. For $n = 1, 2, \dots$, let

$$D_n := \left\{x = (x_1, x_2, \dots, x_d) : x_d > -\frac{1}{n}\right\}$$

and

$$\mathscr{L}_n \equiv \mathscr{L}_n(s, x; \partial_s, \partial_x) := 1_{D_n}(x)\underline{\mathscr{A}}(s, x; \partial_s, \partial_x) + 1_{\partial D_n}(x)\underline{\mathscr{B}}(s, x; \partial_x).$$

Then the parametrix $p_n^{\tau,\eta}(s, x; t, y)$ to the terminal value problems for $\mathscr{L}_n = 0$ on the domain $\overline{D}_n$ is given by

$$p_n^{\tau,\eta}(s, x; t, y) = p_{n,1}^{\tau,\eta}(s, x; t, y) + p_{n,2}^{\tau,\eta}(s, x; t, y).$$

Here

$$\begin{aligned} p_{n,1}^{\tau,\eta}(s, x; t, y) := {} & G(t-s, \underline{\widetilde{U}}^{-1}(\tilde{x} - \tilde{y} - \underline{u}_{dd}^{-1}(x_d - y_d)\underline{\tilde{u}}_d)) \\ & \times \left\{ g(t-s, \underline{u}_{dd}^{-1}(x_d - y_d)) \right. \\ & \left. \qquad - g\left(t-s, \underline{u}_{dd}^{-1}\left(x_d + y_d + \frac{2}{n}\right)\right)\right\} \det \underline{S}^{-1}, \\ p_{n,2}^{\tau,\eta}(s, x; t, y) := {} & 2\underline{\overline{u}}_{dd}^{-1}\int_0^\infty G(t-s, \underline{\widetilde{U}}^{-1}(\tilde{x} - \tilde{y} - \underline{u}_{dd}^{-1}(x_d - y_d + l)\underline{\tilde{u}}_d + l\underline{\mu}'')) \\ & \times h\left(t-s, \underline{u}_{dd}^{-1}\left(x_d + y_d + \frac{2}{n} + l\right)\right) \det \underline{S}^{-1}\, dl, \end{aligned}$$

where $x = (x_1, x_2, \dots, x_d) = (\tilde{x}, x_d)$, $y = (y_1, y_2, \dots, y_d) = (\tilde{y}, y_d)$, and $\underline{\widetilde{U}} = \underline{\widetilde{U}}(\tau, \eta)$, $\underline{\tilde{u}}_d = \underline{\tilde{u}}_d(\tau, \eta)$, etc., are defined from $(\underline{a}_{ij}(\tau, \eta))$ and $(\underline{\mu}_i(\tau, \eta))$ as $\widetilde{U}(\tau, \eta)$, $\tilde{u}_d(\tau, \eta)$, etc., respectively. In the same way as in 2° of §3, we can construct a fundamental solution $p_n(s, x; t, y)$ to the terminal value problem for $\mathscr{L}_n = 0$ on the domain $\overline{D}_n$ and it satisfies the same properties as in Theorem 3.3 with replacing D, $\overline{D}$ and x_d by D_n, $\overline{D}_n$ and $x_d + n^{-1}$, respectively. Moreover we should notice that the constants K and C in Proposition 3.1 and Theorem 3.3 can be chosen so as to be independent of n. Following the construction of the fundamental solutions, we further see that

Theorem 4.1. *For $l = 0, 1$, we have*
(i) $\partial_x^l p_n(s, x; t, y) \to \partial_x^l p(s, x; t, y)$ *as* $n \to \infty$ *for* $0 \le s < t$ *and* $x, y \in \overline{D}$;
(ii) *for any $T > 0$, there exist positive constants K and C such that*

$$|\partial_x^l p_n(s, x; t, y)| \le K(t-s)^{-(d+l)/2}\exp\left\{-C\frac{|x-y|^2}{t-s}\right\}$$

for $0 \le s < t \le T$, $x, y \in \overline{D}$ *and* $n = 1, 2, \dots$.

5. Proof of the main results

1°. *Proof of Theorem* 2.5. The existence of solutions to both the problems is known (see [28], [15]); so we have to prove the uniqueness of solutions and the existence of a transition density. Then we can use the localization argument (see [28, p. 193]; [27, Theorem 3.1, Lemma 3.6, Theorem 3.4]) and hence, in the rest of the proof, we assume that D is the upper half space. Moreover, we may assume that the coefficients b_i of the first order term of $\mathscr{A}_0$ are equal to zero (see [28, Theorem 5.5]).

First we treat the martingale problem. Given any $f \in C_b(R^d)$ and $t > 0$, define

$$v_t(s, x) := \int_{\overline{D}} p(s, x; t, y) f(y)\, dy, \tag{5.1}$$

$$v_t^{(n)}(s, x) := \int_{\overline{D}_n} p_n(s, x; t, y) f(y)\, dy, \tag{5.2}$$

where $p(s, x; t, y)$ and $p_n(s, x; t, y)$ are the fundamental solutions constructed as in §3 on the domains $\overline{D}$ and $\overline{D}_n$ for $\mathscr{L}_0$ and $\mathscr{L}_{n,0}$, respectively. Then, for any $t' \in (0, t)$, $v_t^{(n)} \in C_b^{1,2}([0, t'] \times \overline{D})$ $(n = 1, 2, \dots)$ and, by using Theorem 4.1, we see that

$$|v_t^{(n)}(s, x)| \le K\|f\|_\infty;$$
$$|\partial_x v_t^{(n)}(s, x)| \le K(t-s)^{-1/2}\|f\|_\infty;$$
$$v_t^{(n)}(s, x) \to v_t(s, x) \quad \text{as } n \to \infty \quad \text{for } (s, x) \in [0, t) \times \overline{D};$$
$$\mathscr{B}(s, x; \partial_x) v_t^{(n)}(s, x) \to \mathscr{B}(s, x; \partial_x) v_t(s, x) \quad \text{as } n \to \infty$$

for $(s, x) \in [0, t) \times \partial D$;

$$|\mathscr{B}(s, x; \partial_x) v_t^{(n)}(s, x)| \le K(t - t')^{-1/2}\|f\|_\infty \tag{5.3}$$

for $(s, x) \in [0, t'] \times \partial D$. Let P be any solution of the martingale problem for $\mathscr{L}_0$ starting at $(s, x) \in [0, \infty) \times \overline{D}$. Then, for $s < t' < t$,

$$E[v_t^{(n)}(t', X(t'))] - v_t^{(n)}(s, x) = E\left[\int_s^{t'} \mathscr{B}_0 v_t^{(n)}(u, X(u))\, dl(u)\right],$$

because $\mathscr{A}_0 v_t^{(n)}(u, y) = 0$ for $(u, y) \in [s, t'] \times \overline{D}$. Since $E[l(t')] < \infty$, noting (5.3), we have

$$E\left[\int_s^{t'} \mathscr{B}_0 v_t^{(n)}(u, X(u))\, dl(u)\right] \to 0 \quad \text{as } n \to \infty;$$

hence

$$E[v_t(t', X(t'))] = v_t(s, x). \tag{5.4}$$

We take the limit of the left-hand side of (5.4) with respect to $t' \uparrow t$. Then we see that

$$E[f(X(t))] = v_t(s, x);$$

this yields the uniqueness of solutions to the martingale problem and the existence of a transition density.

Next we show the uniqueness of solutions to the coupled martingale problem. For any $\gamma < 0$, we set

$$\mathscr{B}_\gamma := \mathscr{B}_0 + \gamma,$$

and denote by $p^\gamma(s, x; t, y)$ (resp. $p_n^\gamma(s, x; t, y)$) the fundamental solution constructed as in §3 on $\overline{D}$ (resp. $\overline{D}_n$) for $1_D\mathscr{A}_0 + 1_{\partial D}\mathscr{B}_\gamma$ (resp. $1_{D_n}\mathscr{A}_0 +$

$1_{\partial D_n}\mathscr{B}_\gamma)$. Define $v_t^\gamma(s, x)$ and $v_t^{\gamma,(n)}(s, x)$ as in (5.1) and (5.2) replacing $p(s, x; t, y)$ and $p_n(s, x; t, y)$ by $p^\gamma(s, x; t, y)$ and $p_n^\gamma(s, x; t, y)$, respectively. Let $\overline{P}$ be any solution to the coupled martingale problem for $\mathscr{L}_0$ starting at (s, x). Then, for $s < t' < t$,

$$\overline{E}[v_t^{\gamma,(n)}(t', X(t'))\exp\{\gamma L(t')\}] - v_t^{\gamma,(n)}(s, x) = \overline{E}\left[\int_s^{t'} \mathscr{B}_\gamma v_t^{\gamma,(n)}(u, X(u))\exp\{\gamma L(u)\}\, dL(u)\right],$$

because $\mathscr{A}_0 v_t^{\gamma,(n)}(u, y) = 0$ for $(u, y) \in [s, t'] \times \overline{D}$. In the same way as the case of the martingale problem, we see that

$$\overline{E}[f(X(t))\exp\{\gamma L(t)\}] = v_t^\gamma(s, x) = \int_{\overline{D}} p^\gamma(s, x; t, y) f(y)\, dy;$$

hence each one-dimensional marginal distribution of $\overline{P}$ is uniquely determined and this implies the uniqueness. □

2°. Before proving Theorem 2.8, we shall make a perparatory consideration. We take the defining function Φ of D constructed in the proof of Proposition 2.7. Noting that $\Phi(x) = d(x, \partial D)$ for $x \in \overline{D}$ with $d(x, \partial D) < r/4$, we see that for $x \in \overline{D}$ with $d(x, \partial D) < r/4$ there exists a unique $\overline{x} \in \partial D$ such that $d(x, \partial D) = |x - \overline{x}|$ (see [25, Theorem 1.8]). Therefore we extend the domain of $\beta(s, x)$ in the following way. For $s \geq 0$ and $x \in \overline{D}$ with $d(x, \partial D) < r/4$, we set

$$\overline{\beta}(s, x) = \beta(s, \overline{x}).$$

Then $\overline{\beta}(s, x)$ is bounded continuous in $[0, \infty) \times \{x \in \overline{D}: d(x, \partial D) < r/4\}$. Moreover we see that $\nabla\Phi(x)$ is parallel to $\nabla\Phi(\overline{x})$ and $|\nabla\Phi(x)| = |\nabla\Phi(\overline{x})|$ (cf. [14, Appendix B] or [25, Theorem 1.8]); hence $\overline{\beta}(s, x)\cdot\nabla\Phi(x) = \beta(s, \overline{x})\cdot\nabla\Phi(\overline{x})$. For $k > 0$, we let

$$D_k = \{x \in R^d : \Phi(x) > k\}.$$

Then $D_k \subset D$. Let $s_0 \geq 0$ and $x_0 \in D$. Choose a positive number k_0 such that $k_0 \leq r/4$ and $x_0 \in D_k$ for every $k \in (0, k_0)$. We may assume ∂D_k is contained in the set $\{x \in \overline{D}: d(x, \partial D) < r/4\}$ for every $k \in (0, k_0)$. Thus if we define the operator $\overline{\mathscr{B}}_0 = \overline{\mathscr{B}}_0(s, x; \partial_x)$ on $[0, \infty) \times \partial D_k$ by

$$\overline{\mathscr{B}}_0(s, x; \partial_x) = \overline{\beta}(s, x)\cdot\nabla,$$

then the coefficients of the operators $\mathscr{A}_0$ and $\overline{\mathscr{B}}_0$ are continuous and uniformly bounded in $k \in (0, k_0)$, and further for any $k \in (0, k_0)$

$$\overline{\beta}(s, x)\cdot\nabla\Phi(x) \geq \delta \quad \text{for } (s, x) \in [0, \infty) \times \partial D_k,$$

where δ is the constant in (A.1)(iv). For each $k \in (0, k_0)$ we take a solution $\overline{P}_k$ to the coupled martingale problem on $\overline{D}_k$ for $1_{D_k}\mathscr{A}_0 + 1_{\partial D_k}\overline{\mathscr{B}}_0$ starting at (s_0, x_0). Then the family $\{\overline{P}_k\}$ of solutions is tight, which is proved as in Theorem I.14 in [15]. Therefore there exists a sequence $\{k_n\}$ such that $k_n \downarrow 0$ and $\overline{P}_{k_n}$ converges weakly to a probability measure $\overline{P}$ as $n \to \infty$. Then we have

Lemma 5.1. *Any limit point* $\overline{P}$ *of the family* $\{\overline{P}_k\}$ *is a solution to the coupled martinagle problem on* $\overline{D}$ *for* $\mathscr{L}_0$ *starting at* (s_0, x_0).

Proof. For $f \in C_b^{1,2}([0, \infty) \times R^d)$, we set

$$\overline{M}_f(t) := f(t, X(t)) - f(s_0, x_0) - \int_{s_0}^t \mathscr{A}_0 f(u, X(u))\, du - \int_{s_0}^t \mathscr{B}_0 f(u, X(u))\, dL(u).$$

For notational simplicity, we suppose $\overline{P}_k \to \overline{P}$ weakly as $k \downarrow 0$. Then it is obvious that $\overline{P}[X(s_0) = x_0, L(s_0) = 0] = 1$ and $\overline{P}[\widehat{W} \times \widetilde{V}] = 1$, because $\{X(s_0) = x_0, L(s_0) = 0\}$ and $\widehat{W} \times \widetilde{V}$ are closed sets in U. Now we take a continuous nonnegative function g defined on R^d such that the support is compact and contained in D. Then for sufficiently small k the support of g is contained in D_k. Since

$$\overline{P}_k \left[L(t) = \int_{s_0}^t 1_{\partial D_k}(X(u))\, dL(u) \text{ for } t \geq s_0 \right] = 1,$$

for sufficiently small k

$$\overline{E}_k \left[\int_{s_0}^t g(X(u))\, dL(u) \right] = 0 \quad \text{for } t \geq s_0.$$

Note that for every $t \geq s_0$ the function $(X, L) \in W \times \widetilde{V} \to \int_{s_0}^t g(X(u))\, dL(u)$ is continuous. Next we take a sequence $\{h_m\}$ of bounded continuous functions defined on $[0, \infty)$ such that $h_m(l) = 1$ for $0 \leq l \leq m$, $h_m(l) = 0$ for $l \geq m + 1$ and $0 \leq h_m(l) \leq 1$. Then, for each positive integer m, the function $(X, L) \in W \times \widetilde{V} \to h_m(L(t)) \int_{s_0}^t g(X(u))\, dL(u)$ is bounded and continuous. We extend the domain of this function in such a way that it is bounded continuous on U; this is possible because $W \times \widetilde{V}$ is closed in U. Therefore the equalities $\overline{P}_k[W \times \widetilde{V}] = \overline{P}[W \times \widetilde{V}] = 1$ imply

$$\overline{E} \left[h_m(L(t)) \int_{s_0}^t g(X(u))\, dL(u) \right] = \lim_{k \downarrow 0} \overline{E}_k \left[h_m(L(t)) \int_{s_0}^t g(X(u))\, dL(u) \right] = 0.$$

Letting $m \to \infty$, we have

$$\overline{E} \left[\int_{s_0}^t g(X(u))\, dL(u) \right] = 0 \quad \text{for } t \geq s_0.$$

Therefore

$$\overline{P} \left[\int_{s_0}^t g(X(u))\, dL(u) = 0 \text{ for } t \geq s_0 \right] = 1;$$

hence

$$\overline{P} \left[L(t) = \int_{s_0}^t 1_{\partial D}(X(u))\, dL(u) \text{ for } t \geq s_0 \right] = 1.$$

Now we prove $\overline{E}[L(t)] < +\infty$ for $t \geq s_0$. For the defining function Φ, $\overline{M}_\Phi(t)$ is a $\overline{P}_k$-martingale with respect to the filtration $\{\mathscr{U}_t^{s_0}\}$ for each k and

$$\langle \overline{M}_\Phi \rangle(t) = \int_{s_0}^t a \nabla\Phi \cdot \nabla\Phi(u, X(u))\, du$$

(cf. [15, Theorem I.1]). Noting that

$$\overline{\mathscr{B}}_0\Phi(u, x) = \overline{\beta}(u, x) \cdot \nabla\Phi(x) \geq \delta \quad \text{for } (u, x) \in [s_0, \infty) \times \partial D_k,$$

we see that

$$\begin{aligned}\delta L(t) &\leq \int_{s_0}^t \overline{\mathscr{B}}_0\Phi(u, X(u))\, dL(u) \\ &= \Phi(X(t)) - \Phi(s_0) - \int_{s_0}^t \mathscr{A}_0\Phi(u, X(u))\, du - \overline{M}_\Phi(t), \quad \overline{P}_k\text{-a.s.};\end{aligned}$$

hence

$$\begin{aligned}\delta^2\overline{E}_k[L(t)^2] \leq 4\Bigg\{&\overline{E}_k[\Phi(X(t))^2] + \overline{E}_k[\Phi(X(s_0))^2] \\ &+ \overline{E}_k\left[\left(\int_{s_0}^t \mathscr{A}_0\Phi(u, X(u))\, du\right)^2\right] + \overline{E}_k[\langle\overline{M}_\Phi\rangle(t)]\Bigg\}.\end{aligned}$$

The boundedness of the coefficients of $\mathscr{A}_0$ and the fact $\Phi \in C_b^2(R^d)$ imply

$$\sup_k \overline{E}_k[L(t)^2] < +\infty.$$

Since $h_m(L(t))L(t)^2$ is bounded continuous in L of $\widetilde{V}$,

$$\overline{E}[h_m(L(t))L(t)^2] = \lim_{k\downarrow 0} \overline{E}_k[h_m(L(t))L(t)^2] \leq \sup_k \overline{E}_k[L(t)^2].$$

Letting $m \to \infty$, we have $\overline{E}[L(t)^2] < +\infty$; hence, $\overline{E}[L(t)] < +\infty$. Finally we shall show that $\overline{M}_f(t)$ is a $\overline{P}$-martingale. We note that, for each $t \geq s_0$, $\overline{M}_f(t)$ is continuous in (X, L) of $W \times \widetilde{V}$ and, for $s_0 \leq t < t'$,

$$h_m(L(t'))\{\overline{M}_f(t') - \overline{M}_f(t)\} \qquad (m = 1, 2, \dots)$$

are bounded continuous in (X, L) of $W \times \widetilde{V}$. For $s_0 \leq t_0 \leq \cdots \leq t_n \leq t < t'$ and bounded continuous functions $g_0(x, l), \dots, g_n(x, l)$, define

$$G(X, L) := g_0(X(t_0), L(t_0)) \cdots g_n(X(t_n), L(t_n));$$

then

$$\overline{E}_k[\{\overline{M}_f(t') - \overline{M}_f(t)\}G(X, L)] = 0.$$

Therefore

$$\begin{aligned}&|\overline{E}_k[h_m(L(t'))\{\overline{M}_f(t') - \overline{M}_f(t)\}G(X, L)]| \\ &\quad = |\overline{E}_k[\{h_m(L(t')) - 1\}\{\overline{M}_f(t') - \overline{M}_f(t)\}G(X, L)]| \\ &\quad \leq K\|G\|_\infty\{\|f\|_\infty + \|\partial_x f\|_\infty + \|\partial_x^2 f\|_\infty\}\overline{E}_k[\{1 - h_m(L(t'))\}] \\ &\qquad + K\|G\|_\infty\|\partial_x f\|_\infty\overline{E}_k[\{1 - h_m(L(t'))\}L(t')] \\ &\quad \leq K_1\overline{P}_k[L(t') \geq m] + K_2\overline{E}_k[L(t'); L(t') \geq m],\end{aligned}$$

where we set

$$K_1 = K\|G\|_\infty\{\|f\|_\infty + \|\partial_x f\|_\infty + \|\partial_x^2 f\|_\infty\} \quad \text{and} \quad K_2 = K\|G\|_\infty\|\partial_x f\|_\infty.$$

This yields that

$$\begin{aligned} &|\overline{E}_k[h_m(L(t'))\{\overline{M}_f(t') - \overline{M}_f(t)\}G(X, L)]| \\ &\qquad \le K_1 \frac{1}{m^2} \sup_k \overline{E}_k[L(t')^2] + K_2 \frac{1}{m} \sup_k \overline{E}_k[L(t')^2]. \end{aligned}$$

Since the left-hand side of the inequality above converges to

$$|\overline{E}[h_m(L(t'))\{\overline{M}_f(t') - \overline{M}_f(t)\}G(X, L)]| \quad \text{as } k \downarrow 0,$$

tending m to infinity, we have

$$\overline{E}[\{\overline{M}_f(t') - \overline{M}_f(t)\}G(X, L)] = 0;$$

that is, $\{\overline{M}_f(t)\}$ is a $\overline{P}$-martingale. This completes the proof. □

Proof of Theorem 2.8. For fixed $s \ge 0$ and $x \in D$, we choose a positive number k_0 such that $x \in D_{k_0}$, and for each $k \in (0, k_0)$ we take a solution $\overline{P}_k$ to the coupled martingale problem on $\overline{D}_k$ for $1_{D_k}\mathscr{A}_0 + 1_{\partial D_k}\overline{\mathscr{B}}_0$ starting at (s, x). Then, by Theorem 2.5 and the result described above, it follows that $\overline{P}_k$ converges to the unique solution $\overline{P}_{s,x}$ to the coupled martingale problem on $\overline{D}$ for $\mathscr{L}_0$ starting at (s, x) as $k \downarrow 0$. Now we take a fundamental solution $p(s, x; t, y)$ to the terminal value problem for $\mathscr{L} = 0$ and set, for $t > 0$,

$$v_t(s, x) := \int_{\overline{D}} p(s, x; t, y) f(y)\, dy.$$

Then, for $s < t' < t$,

$$\begin{aligned} &\overline{E}_k\left[v_t(t', X(t')) \exp\left\{\int_s^{t'} c(u, X(u))\, du \right.\right. \\ &\qquad\qquad \left.\left. + \int_s^{t'} \gamma(u, X(u))\, dL(u)\right\}\right] - v_t(s, x) \qquad (5.5)\\ &\quad = \overline{E}_k\left[\int_s^{t'} \overline{\mathscr{B}}_0 v_t(u, X(u)) \exp\left\{\int_s^u \gamma(\tau, X(\tau))\, dL(\tau)\right\} dL(u)\right]. \end{aligned}$$

When k tends to zero, the expectation of the left-hand side converges to the expectation of the same integrand with respect to $\overline{P}_{s,x}$. The right-hand side of (5.5) is estimated as follows:

$$\begin{aligned}
&\left|\overline{E}_k\left[\int_s^{t'} \overline{\mathscr{B}}_0 v_t(u, X(u)) \exp\left\{\int_s^u \gamma(\tau, X(\tau))\, dL(\tau)\right\} dL(u)\right]\right| \\
&\quad \le \overline{E}_k\left[\int_s^{t'} |\overline{\mathscr{B}}_0 v_t(u, X(u))|\, dL(u)\right] \\
&\quad \le \overline{E}_k\left[h_m(L(t'))\int_s^{t'} |\overline{\mathscr{B}}_0 v_t(u, X(u))|\, dL(u)\right] \\
&\qquad + \overline{E}_k\left[\{1 - h_m(L(t'))\}\int_s^{t'} |\overline{\mathscr{B}}_0 v_t(u, X(u))|\, dL(u)\right] \\
&\quad \le \overline{E}_k\left[h_m(L(t'))\int_s^{t'} |\overline{\mathscr{B}}_0 v_t(u, X(u))|\, dL(u)\right] \\
&\qquad + K\|\partial_x v_t\|_\infty \overline{E}_k[L(t')\,;\, L(t') \ge m] \\
&\quad \le \overline{E}_k\left[h_m(L(t'))\int_s^{t'} |\overline{\mathscr{B}}_0 v_t(u, X(u))|\, dL(u)\right] \\
&\qquad + K\|\partial_x v_t\|_\infty \frac{1}{m}\sup_k \overline{E}_k[L(t')^2].
\end{aligned} \tag{5.6}$$

The first term of the last right-hand side of (5.6) converges to

$$\overline{E}_{s,x}\left[h_m(L(t'))\int_s^{t'} |\overline{\mathscr{B}}_0 v_t(u, X(u))|\, dL(u)\right]$$

as $k \downarrow 0$ and the limit is equal to zero because $\overline{\mathscr{B}}_0 v_t(u, y) = 0$ for $(u, y) \in [s, t) \times \partial D$, and the second term of the last right-hand side of (5.6) converges to zero as $m \to \infty$. That is,

$$\begin{aligned}
&\overline{E}_{s,x}\left[v_t(t', X(t')) \exp\left\{\int_s^{t'} c(u, X(u))\, du + \int_s^{t'} \gamma(u, X(u))\, dL(u)\right\}\right] \\
&\quad = v_t(s, x);
\end{aligned}$$

hence, then $t' \uparrow t$, we have

$$\overline{E}_{s,x}\left[v_t(t, X(t)) \exp\left\{\int_s^t c(u, X(u))\, du + \int_s^t \gamma(u, X(u))\, dL(u)\right\}\right] = v_t(s, x).$$

This proves the assertion when $x \in D$. If $x \in \partial D$, we take a sequence $\{x_n\}$ such that $x_n \in D$ and $x_n \to x$ as $n \to \infty$. Then $\overline{P}_{s,x_n}$ converges to $\overline{P}_{s,x}$ weakly as $n \to \infty$. Therefore we also have the same conclusion for $x \in \partial D$. □

Remark. From the proof of Theorem 2.8, we see that, under the assumption (A-2), the conclusion of Theorem 2.8 is essentially due to the uniqueness of solutions to the coupled martingale problem.

6. Additional remarks

1°. If the Hölder exponent α of the coefficients of $\mathscr{L}$ is greater than $1/2$, then it is possible to construct a fundamental solution to the terminal value problem for $\mathscr{L} = 0$ on the upper half space by applying the parametrix method once (see [34]).

2°. By using the parametrix given in §3 and by applying the parametrix method twice as in §3, a fundamental solution to the terminal value problem for $\mathscr{L} = 0$ on a smooth compact domain in a Riemannian manifold is constructed in [17].

3°. Under the same situation as in Theorem 2.5, by a method similar to [37], we can verify the uniqueness of invariant probability measures for the diffusion process.

4°. In contrast with the case where the coefficients β_i of the boundary operator $\mathscr{B}$ are smoother (see [8], [11]), $\mathscr{L}$ has no fundamental solutions with C^2-smoothness up to the boundary in general. We shall give such an example. Let us take D, $\mathscr{A}$ and $\mathscr{B}$ as follows:

$$D = \{x = (x_1, x_2) : x_2 > 0\} \quad \text{(the upper half plane)},$$
$$\mathscr{A} = \frac{\partial}{\partial s} + \frac{1}{2}\Delta, \quad \mathscr{B} = \mu(x_1)\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},$$

where $\mu(x_1)$ is a bounded, positive, α-Hölder continuous, nowhere differentiable function $(0 < \alpha < 1)$. Then the fundamental solution $p(s, x; t, y)$ to the terminal value problem for $\mathscr{L} = 0$ does not have the C^2-smoothness up to the boundary. First, we note that, for each $t > 0$ and $y \in \overline{D}$, $\frac{\partial p}{\partial x_2}(\cdot, \cdot; t, y) \not\equiv 0$ on $[0, t) \times \partial D$. In fact, if $\frac{\partial p}{\partial x_2}(\cdot, \cdot; t, y) \equiv 0$ on $[0, t) \times \partial D$, by the uniqueness result (Theorem 2.8, Remark), $p(s, x; t, y)$ is the fundamental solution to the terminal value problem for $1_D\mathscr{A} + 1_{\partial D}\mathscr{B}_0 = 0$, where $\mathscr{B}_0 = \frac{\partial}{\partial x_2}$; that is,

$$p(s, x; t, y) = g(t - s, x_1 - y_1)\{g(t - s, x_2 - y_2) + g(t - s, x_2 + y_2)\}.$$

Therefore

$$\frac{\partial p}{\partial x_1}(s, x; t, y) \neq 0 \quad \text{for } (s, x) \in [0, t) \times \partial D \text{ with } x_1 \neq y_1;$$

this leads to a contradiction. Hence there exists a point $(s_0, x_0) \in [0, t) \times \partial D$ such that $\frac{\partial p}{\partial x_2}(s_0, x_0; t, y) \neq 0$ (hence, $\frac{\partial p}{\partial x_1}(s_0, x_0; t, y) \neq 0$); then for $x = (x_1, 0)$ near x_0

$$\mu(x_1) = -\frac{\partial p}{\partial x_2}(s_0, x; t, y) / \frac{\partial p}{\partial x_1}(s_0, x; t, y).$$

The condition $p(s_0, \cdot; t, y) \in C^2(\overline{D})$ implies that $\mu(x_1)$ is differentiable; this contradicts the assumption on $\mu(x_1)$.

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